Quantum mechanics differential equations and the de Broglie wavelength.

Osvaldo Domann

odomann@yahoo.com

Last revision August 2014

(This work is an extract of [4] listed in sec. References.)

Abstract

Quantum mechanics differential equations are based on the de Broglie wavelength assigned to a particle.

This paper presents the effect on quantum mechanics differential equations when replacing the de Broglie wavelength by a relation between the radius and the total energy of a particle. This relation results from a theoretical work of the author about the interaction of charged particles, where particles are modelled as emitting and absorbing continuously fundamental particles with longitudinal and transversal angular momentum. The laws of interaction are mathematically formulated and then proven in that the basic laws of physics (Coulomb, Ampere, Lorentz, Maxwell, Gravitation, bending and interference of light and particles), are deduced from them.

The main effect on quantum mechanics is the change of the pair of canonical conjugated variables that are linked by the uncertainty principle of Heisenberg, namely, to the pairs **energy-space** and **momentum-time**. This has the consequence on relativistic and non relativistic operators in that they are reversed respect to time and space derivation compared with the standard theory.

The accordance of the proposed theory with the correspondence principle of quantum mechanics is proven, in that the time independent differential equation from Schroedinger, deduced from the wave package constructed with the de Broglie wavelength, can be derived from the wave package constructed with the radius-energy relation.

Solutions of the new differential equations for the potential pot, the harmonic oscillator and the hydrogen atom are presented and compared with the solutions of the Schroedinger differential equations.

1 Introduction.

Quantum mechanics differential equations are based on the de Broglie wavelength. In the original work about the interaction of charged particles [4], this particles are represented by a non-local model emitting and absorbing continuously fundamental particles. The energy of a particle is distributed in space from r_o to infinity, where r_o is the radius of the particle. The following relation between the radius r_o and the energy of a charged particle is derived .

$$r_o = \frac{\hbar\, c}{E} \qquad with \qquad E_{rel} = \sqrt{E_o^2 + E_p^2} \;\; the \;\; relativistic \;\; energy. \tag{1}$$

This relation is used instead of the de Broglie wavelength in order to build wave packages with a Gauss distribution and to derive the corresponding probability differential equations of quantum mechanics. The effects on the uncertainty relations and the main quantum mechanics operators are presented.

Note: When deriving the wave-packge with the radius-energy relation, the mass of a particle is considered as concentrated in a sphere with a diameter equal approximately to two times the radius r_o given by the radius energy-relation. This is not according to the approach [4] that let to the radius-energy relation where the mass (energy) of a particle is distributed from r_o to infinity, outside the sphere with radius r_o .

2 General considerations.

To make use of the of Fourier-Transformation, the movement of a particle is first described as a sequence of particles represented by a sinus wave, having a wavelength λ equal to $2\pi r_o$. Then the Fourier-Transformation of a wave package of sinus waves with a Gauss shaped amplitude is build.

We have that

$$\lambda = 2\pi r_o = 2\pi \frac{\hbar\, c}{E_{rel}} \qquad with \qquad E_{rel} = \sqrt{E_o^2 + E_p^2} \tag{2}$$

or

$$E_{rel} = E_o + E_{kin} = E_o + E_o \left[\frac{1}{2}\,\mu - \frac{1\cdot 1}{2\cdot 4}\,\mu^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\,\mu^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\,\mu^4 + \cdots \right] \tag{3}$$

with

$$E_o = m_o\, c^2 \qquad E_p = p\, c \qquad p = \frac{m_o\, c}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \mu = \frac{E_p^2}{E_o^2} \tag{4}$$

The sinus wave on the x-axis is

$$\xi_x = A\, e^{i(k_x\, x - \omega_x\, t)} \qquad with \qquad k_x = \frac{2\pi}{\lambda_x} \qquad and \qquad \omega_x = 2\pi\, \frac{v_x}{\lambda_x} \tag{5}$$

If we now introduce in the expression that $\lambda_x = 2\pi r_{o_x} = 2\pi\hbar c / E_{rel_x}$ we get

$$\xi_x = A\, exp \left[i\frac{c}{\hbar} \left(\frac{E_{rel_x}}{c^2}\, x - \frac{v_x}{c^2}\, E_{rel_x}\, t \right) \right] \tag{6}$$

or

$$\xi_x = A\, exp \left[i\frac{c}{\hbar} \left(\frac{E_{rel_x}}{c^2}\, x - \, p_x\, t \right) \right] \tag{7}$$

with

$$E_{rel_x} = E_o + E_{kin_x} = m_o\, c^2 \left(1 - \frac{v_x^2}{c^2} \right)^{-1/2} \qquad and \qquad p_x = \frac{v_x}{c^2}\, E_{rel_x} \tag{8}$$

with E_{kin_x} the relativistic kinetic energy of the particle on the x-axis.

3 The wave package.

We define the Fourier-Transformation of a wave package [1,2]; on the x-axis as

$$\phi_x(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \kappa_x(p_x)\, exp \left\{ i\frac{c}{\hbar} \left[m_{rel_x}(p_x)\, x - \, p_x\, t \right] \right\} dp_x \tag{9}$$

with a Gauss distribution $\kappa_x(p_x)$ on the p_x-axis

$$\kappa_x(p_x) = B\, exp \left\{ - \frac{(p_x - p_{x_o})^2}{4(\Delta p_x)^2} \right\} \tag{10}$$

and the dispersion $m_{rel_x} = m_{rel_x}(p_x)$ with

$$m_{rel_x} = \frac{E_{rel_x}}{c^2} \qquad m_{rel_x} = m_{rel_x}(p_x) = \frac{1}{c^2} \sqrt{E_o^2 + p_x^2 c^2} \qquad and \qquad E_o = m_o c^2 \tag{11}$$

Because of symmetry reasons we also have a wave package

$$\psi_x(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi_x(m_{rel_x})\, exp \left\{ i\frac{c}{\hbar} \left[m_{rel_x}\, x - \, p_x(m_{rel_x})\, t \right] \right\} dm_{rel_x} \tag{12}$$

with the Gauss distribution on the m_{rel_x}-axis

$$\chi_x(m_{rel_x}) = A\ exp\ \left\{ -\frac{(m_{rel_x} - m_{rel_{x_o}})^2}{4(\Delta m_{rel_x})^2} \right\} \tag{13}$$

and the dispersion

$$p_x(m_{rel_x}) = c\ \sqrt{m_{rel_x}^2\ -\ m_o^2} \qquad and \qquad m_o = \frac{E_o}{c^2} \tag{14}$$

4 Probability differential equations.

In this section the probability differential equations are derived. The differential equations are classified into unrestricted and non-relativistic. Then they are subclassified in groups of general, time or space independent.

4.1 Unrestricted differential equations.

The unrestricted differential equations are valid for the whole range of speed $0 \leq v \leq c$.

We start with the wave package

$$\psi_x(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi_x(m_{rel_x})\ exp \left\{ i\frac{c}{\hbar} \left[m_{rel_x}\ x -\ p_x(m_{rel_x})\ t \right] \right\} dm_{rel_x} \tag{15}$$

with

$$m_{rel_x} = \frac{E_{rel_x}}{c^2} \qquad and \qquad p_x(m_{rel_x}) = c\ \sqrt{m_{rel_x}^2 - m_o^2} \tag{16}$$

with

$$E_{rel_x} = E_o + E_{kin_x} = \sqrt{E_o^2 + E_{p_x}^2} \qquad E_o = m_o c^2 \qquad E_{p_x} = p_x\ c \tag{17}$$

For the unrestricted range of velocities $0 \leq v \leq c$ we have that

$$p_x = \frac{v_x}{c^2} E_{rel_x} \tag{18}$$

and E_{kin_x} represents the kinetic energy for the whole range of speed.

4.1.1 The wave equation.

The wave differential equation we obtain by derivation of ψ_x two times versus t and two times versus x. The results are then connected through

$$p_x = \frac{v_x}{c^2} E_{rel_x} \tag{19}$$

We get

$$\frac{\partial^2}{\partial x^2}\psi_x \;=\; \frac{1}{v_x^2}\,\frac{\partial^2}{\partial t^2}\psi_x \tag{20}$$

For $v_x \to c$ we have

$$\frac{\partial^2}{\partial x^2}\psi_x(x,t) \;=\; \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\psi_x(x,t) \tag{21}$$

the well known wave equation

4.1.2 The time independent differential equation.

Time independent differential equations are deduced derivating one time and two times the wave function ψ_x.

a) We derivate the wave function ψ_x one time versus x and get the following time independent differential equation on the x coordinate

$$\frac{\partial}{\partial x}\psi_x = \frac{i}{\hbar\, c}\; E_{rel_x}\;\psi_x = \frac{i}{\hbar\, c}\;(E_o + E_{kin_x})\;\psi_x \tag{22}$$

E_{kin_x} represents the kinetic energy for the whole range of speed, relativistic and non-relativistic.

b) We derivate the wave function ψ_x two times versus x and get the following time independent differential equation on the x coordinate

$$\frac{\partial^2}{\partial x^2}\psi_x = -\,\frac{c^2}{\hbar^2}\; m_{rel_x}^2\;\psi_x \tag{23}$$

With

$$m_{rel_x} = \frac{1}{c^2}\sqrt{E_o^2 + E_{p_x}^2} \qquad\qquad E_o = m_o c^2 \qquad \mathit{and} \qquad E_{p_x} = p_x\; c \tag{24}$$

we get

$$\frac{\partial^2}{\partial x^2}\psi_x = -\,\frac{1}{\hbar^2\; c^2}\;(E_o^2 + E_{p_x}^2)\;\psi_x \tag{25}$$

4.1.3 The space independent differential equation.

We derivate the wave function ψ_x two times versus t

$$\frac{\partial^2}{\partial t^2}\psi_x = -\,\frac{c^2}{\hbar^2}\;p_x^2\;\psi_x \tag{26}$$

and with

$$E_{p_x} = p_x\, c \qquad and \qquad E_p^2 = E_{p_x}^2 + E_{p_y}^2 + E_{p_z}^2 \tag{27}$$

we get

$$-\,\hbar^2\, \frac{\partial^2}{\partial t^2}\psi_x = E_{p_x}^2 \psi_x \tag{28}$$

and for the space

$$-\,\hbar^2\, \Delta_{\mathbf{t}}\psi = E_p^2 \psi \tag{29}$$

with the operator $\Delta_{\mathbf{t}}$ defined in sec. 6.

4.2 Non relativistic differential equations

For non relativistic speeds we have that $v \ll c$ and that $E_{kin_x} \approx p^2/(2m_o)$.

4.2.1 General non relativistic differential equation.

The general non relativistic differential equation we obtain by deriving ψ_x two times versus t and one time versus x. The results are then connected through $E_{kin_x} \approx p^2/(2m_o)$. We get

$$-\,i\,\hbar\, c\frac{\partial}{\partial x}\psi_x(x,t) \;-\; E_o\,\psi_x(x,t) \approx \;-\;\frac{\hbar^2}{2\,m_o\,c^2}\,\frac{\partial^2}{\partial t^2}\psi_x(x,t) \quad with \quad E_o = m_o\,c^2 \tag{30}$$

The differential equation with the constant energy E_o describes the movement of a non-accelerated particle in a cero potential energy field.

With E_{tot} the total energy, E_{kin} the kinetic energy, E_{pot} the potential energy and E_{rel} the relativistic energy, the above equation is equivalent to $E_{rel} \;-\; E_o = E_{kin}$. If we add at both sides the potential energy $E_{pot} = U_x(x,t)$ we get the equations for an accelerated movement. The result is

$$-\,i\,\hbar\, c\frac{\partial}{\partial x}\psi_x(x,t) \;-\; E_o\,\psi_x(x,t) \;+\; U_x(x,t)\psi_x(x,t) = E_{tot}\psi_x(x,t) \tag{31}$$

$$-\;\frac{\hbar^2}{2\,m_o\,c^2}\,\frac{\partial^2}{\partial t^2}\psi_x(x,t) \;+\; U_x(x,t)\psi_x(x,t) = E_{tot}\psi_x(x,t) \tag{32}$$

Comparing equation (30) with the **General Schrödinger** differential equation, the main difference is that equation (30) derives one time versus space and two times versus time, in other words, time and space are interchanged.

4.2.2 The time independent non relativistic differential equation.

Differential equations are deduced in derivating one time or two times the wave function ψ_x.

a) We derivate the wave function ψ_x one time versus x

$$\frac{\partial}{\partial x}\psi_x = \frac{i}{\hbar\, c}\; E_{rel_x}\; \psi_x = \frac{i}{\hbar\, c}\; (E_o + E_{kin_x})\; \psi_x \tag{33}$$

For a conservative field $U_x = q_e\; V_x$ with a total energy E_{tot_x} we have

$$E_{tot_x} = E_{kin_x} + U_x \qquad and\ \ with \qquad E_{kin_x} \approx \frac{1}{2\; m_o}\; p_x^2 \tag{34}$$

we get

$$\left\{ -\; i\; \hbar\; c\; \frac{\partial}{\partial x} + U(x) \right\} \psi(x) \approx E_x\; \psi(x) \tag{35}$$

with

$$E_x = E_{tot_x} + E_o \tag{36}$$

the Eigenvalue.

b) For the time independent differential equation deduced derivating the wave function ψ_x two times versus x see sec. 7.

4.2.3 Space independent non relativistic differential equation.

We take two times the derivate of the wave function ψ_x versus t

$$\frac{\partial^2}{\partial t^2}\psi_x = -\; \frac{c^2}{\hbar^2}\; p_x^2\; \psi_x \tag{37}$$

and with eq. (29)

$$-\; \hbar^2\; \Delta_{\mathbf{t}}\psi = E_p^2\psi \tag{38}$$

and $v \ll c$ and a conservative potential U

$$E_{kin} \approx \frac{1}{2\; m_o}\; p^2 = \frac{E_p^2}{2\; E_o} \qquad and \qquad E_{tot} = E_{kin} + U \tag{39}$$

we obtain the space independent non relativistic differential equation

$$\left\{ -\; \frac{\hbar^2}{2\; E_o}\; \Delta_{\mathbf{t}} + U \right\} \psi \approx E_{tot}\; \psi \tag{40}$$

which is equivalent to the time inependent equation from Schroedinger.

5 Uncertainty principle.

In the proposed model the pairs of canonical conjugated variables lead to the following uncertainty relations

$$(\Delta E) \cdot (\Delta x) \geq \frac{1}{2}\, \hbar\, c \tag{41}$$

and

$$(\Delta p) \cdot (\Delta t) \geq \frac{1}{2}\, \frac{\hbar}{c} \tag{42}$$

Noticeable at this point is the relation

$$E\, r_o = \hbar\, c \tag{43}$$

for a particle, that connects the radius r_o and the relativistic energy E through $\hbar\, c$.

6 Operators.

6.1 Relativistic operators.

6.1.1 Relativistic operator for the linear momentum.

The relativistic operator for the linear momentum of a particle is

$$\hat{p} = \; i\, \frac{\hbar}{c}\, \frac{\partial}{\partial t} \tag{44}$$

The linear momentum we get with

$$\bar{p}\, \chi = \; i\, \frac{\hbar}{c}\, \nabla_t\, \chi \tag{45}$$

where χ is the total mass-probability function

$$\chi = \psi_x\; \psi_y\; \psi_z \tag{46}$$

and $\nabla_{\mathbf{t}}$

$$\nabla_{\mathbf{t}} = \; \frac{\partial}{\partial t}|_x\; \mathbf{e}_x \; + \; \frac{\partial}{\partial t}|_y\, \mathbf{e}_y \; + \; \frac{\partial}{\partial t}|_z\; \mathbf{e}_z \tag{47}$$

6.1.2 Relativistic operators for the energy.

For the relativistic energy of a non-accelerated particle we obtain the operator

$$\hat{E}_{rel_x} = -\, i\, \hbar\, c\, \frac{\partial}{\partial x} \tag{48}$$

Application example.
If we apply the relativistic operators to the relativistic energy of a particle

$$E_x^2 = m_o^2\, c^4\; +\; p_x^2\, c^2 \tag{49}$$

we get

$$-\, \hbar^2\, c^2\, \frac{\partial^2}{\partial x^2} \psi_x = m_o^2\, c^4\, \psi_x\; -\; \hbar^2\, \frac{\partial^2}{\partial t^2} \psi_x \tag{50}$$

the **Klein-Gordon** equation.
With $m_o = 0$ we have

$$\frac{\partial^2}{\partial x^2} \psi_x\; =\; \frac{1}{c^2}\, \frac{\partial^2}{\partial t^2} \psi_x \tag{51}$$

6.2 Non-relativistic operators.

6.2.1 Non-relativistic operator for the kinetic energy.

The non-relativistic operator for the kinetic energy on the x coordinate is

$$\hat{E}_{kin_x} = -\, \frac{\hbar^2}{2\, m_o\, c^2}\, \frac{\partial^2}{\partial t^2}|_x \tag{52}$$

and the total kinetic energy E_{kin} in the three dimensional space

$$E_{kin} = E_{kin_x} + E_{kin_y} + E_{kin_z} = -\, \frac{\hbar^2}{2\, m_o\, c^2}\, \Delta_{\mathbf{t}} \chi \tag{53}$$

with

$$\Delta_{\mathbf{t}} = \frac{\partial^2}{\partial t^2}|_x\; +\; \frac{\partial^2}{\partial t^2}|_y\; +\; \frac{\partial^2}{\partial t^2}|_z \tag{54}$$

6.2.2 Non-relativistic Hamilton operator.

The operator for the non-relativistic total energy on the x coordinate has the form

$$\hat{E}_x = \frac{1}{2\, m_o} \left(i\, \frac{\hbar}{c}\, \frac{\partial}{\partial t}|_x \right)^2\; +\; \hat{U}_x \tag{55}$$

or

$$\hat{E}_x = \frac{\hat{p}_x^{\,2}}{2\; m_o} \; + \; \hat{U}_x \tag{56}$$

which is equal to the Hamilton operator $\hat{H}_x$.
The general non-relativistic differential equation thus takes the form

$$i\; \hbar c\; \frac{\partial}{\partial x} \psi_x(x,t) = \hat{H}_x\; \psi_x(x,t) \tag{57}$$

with

$$\hat{H}_x = \frac{\hat{p}_x^{\,2}}{2\; m_o} \; + \; \hat{U}_x \tag{58}$$

the non-relativistic Hamilton operator.

6.2.3 Non-relativistic operator for the orbital-angular-momentum.

The wave function for the three dimentional space is

$$\psi_x(\mathbf{r},t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(m_{rel})\; exp\left\{ i \frac{c}{\hbar} \left[m_{rel}\; \mathbf{r} - \; \mathbf{p}(m_{rel})\; t \right] \right\} dm_{rel} \tag{59}$$

with

$$\mathbf{r} = x\; \mathbf{e}_x + y\; \mathbf{e}_y + z\; \mathbf{e}_z \qquad and \qquad \mathbf{p} = p_x\; \mathbf{e}_x + p_y\; \mathbf{e}_y + p\; \mathbf{e}_z \tag{60}$$

We define the linear momentum operator for the different coordinates as:

$$\hat{p}_k = \; i\; \frac{\hbar}{c}\; \frac{\partial}{\partial t}|_k \tag{61}$$

The orbital-angular-momentum-operator can be expressed as

$$\mathbf{M} \left(\mathbf{r},\; i\; \frac{\hbar}{c}\; \nabla_{\mathbf{t}} \right) = \left(\mathbf{r}\; \times\; i\; \frac{\hbar}{c}\; \nabla_{\mathbf{t}} \right) \tag{62}$$

with

$$\nabla_{\mathbf{t}} = \; \frac{\partial}{\partial t}|_x\; \mathbf{e}_x \; + \; \frac{\partial}{\partial t}|_y\; \mathbf{e}_y \; + \; \frac{\partial}{\partial t}|_z\; \mathbf{e}_z \tag{63}$$

The operators for the vectorcomponents are:

$$\hat{M}_x = \hat{y}\; \hat{p}_z - \hat{z}\; \hat{p}_y \qquad \hat{M}_y = \hat{z}\; \hat{p}_x - \hat{x}\; \hat{p}_z \qquad \hat{M}_z = \hat{x}\; \hat{p}_y - \hat{y}\; \hat{p}_z \tag{64}$$

The conmutations are as known

$$[\hat{M}_k, \hat{M}_{k+1}] \neq 0 \qquad [\hat{M}_k, \hat{Q}] = 0 \qquad with \qquad \hat{Q} = \hat{M}_x^2 + \hat{M}_y^2 + \hat{M}_z^2 \tag{65}$$

7 The proposed theory and the Correspondence Principle.

The present theory is based on the radius-energy relation derived in [4], relation that substitutes the de Broglie wavelength.

The accordance of the proposed theory with the correspondence principle of the older quantum mechanics is ensured, in that the time independent differential equation from Schroedinger, deduced from the wave package constructed with the de Broglie wavelength, can be derived from the wave package constructed with the radius-energy relation presented in [4].

We start derivating the wave function ψ_x two times versus space, to get the time independent differential equation

$$\frac{\partial^2}{\partial x^2}\psi_x = -\,\frac{c^2}{\hbar^2}\,m_{rel_x}^2\,\psi_x \tag{66}$$

With

$$m_{rel_x} = \frac{1}{c^2}\sqrt{E_o^2 + E_{p_x}^2} \qquad E_o = m_o c^2 \qquad and \qquad E_{p_x} = p_x\,c \tag{67}$$

we get

$$\frac{\partial^2}{\partial x^2}\psi_x = -\,\frac{1}{\hbar^2\,c^2}\,(E_o^2 + E_{p_x}^2)\,\psi_x \tag{68}$$

For non-relativistic velocities $v \ll c$ we have that

$$E_{kin_x} = \frac{p_x^2}{2\,m_o} \qquad and \qquad E_{p_x}^2 = p_x^2\,c^2 = 2\,m_o\,c^2\,E_{kin_x} \tag{69}$$

and we get

$$\frac{\partial^2}{\partial x^2}\psi_x = -\,\frac{2\,m_o}{\hbar^2}\,\left[\frac{1}{2}E_o + E_{kin_x}\right]\,\psi_x \tag{70}$$

With a conservative potential $E_{tot_x} = U_x + E_{kin_x}$ we get finally

$$\left[-\,\frac{\hbar^2}{2\,m_o}\frac{\partial^2}{\partial x^2}\,+\,U_x\right]\psi_x \;=\; E_x\,\psi_x \qquad with \qquad E_x = \frac{1}{2}\,[E_o + 2\,E_{tot_x}] \tag{71}$$

For the three dimensional space we have

$$\left[- \frac{\hbar^2}{2\, m_o}\, \Delta_{\mathbf{r}} \;+\; U\right]\, \chi = E\, \chi \tag{72}$$

with $\Delta_{\mathbf{r}}$ the Laplace operator and

$$E = \frac{1}{2}\,[E_o + 2\, E_{tot}] \tag{73}$$

If we make $E_o = 0$ we get

$$\left[- \frac{\hbar^2}{2\, m_o}\, \Delta_{\mathbf{r}} \;+\; U\right]\, \chi = E_{tot}\, \chi \tag{74}$$

Eq. (74) is exactly the time independent differential equation constructed by **Schroedinger** with E_{tot} the Eigenvalue.

8 The mass conservation equation.

The mass conservation differential equation we obtain by derivating ψ_x one time versus t and one time versus x. The results are then connected through

$$p_x = \frac{v_x}{c^2} E_{rel_x} \tag{75}$$

We get

$$\frac{\partial}{\partial t}\psi_x(x,t) \;\;=\;\; -\, v_x\, \frac{\partial}{\partial x}\psi_x(x,t) \tag{76}$$

We define the mass probability density as

$$\rho_x(x,t) = \psi_x^*(x,t)\; \psi_x(x,t) \qquad or \qquad \rho(\mathbf{r},t) = \psi^*(\mathbf{r},t)\; \psi(\mathbf{r},t) \tag{77}$$

We derive the mass probability density versus time

$$\frac{\partial}{\partial t}\rho_x(x,t) = \frac{\partial}{\partial t}\,[\psi_x^*(x,t)\; \psi_x(x,t)] = \frac{\partial}{\partial t}\psi_x^*(x,t)\; \psi_x(x,t) \;+\; \psi_x^*(x,t)\, \frac{\partial}{\partial t}\psi_x(x,t) \tag{78}$$

With eq. (76) we get

$$\frac{\partial}{\partial t}\rho_x(x,t) = -v_x \left[\frac{\partial}{\partial x}\psi_x^*(x,t)\; \psi_x(x,t) \;+\; \psi_x^*(x,t)\, \frac{\partial}{\partial x}\psi_x(x,t)\right] \tag{79}$$

or

$$\frac{\partial}{\partial t}\rho_x(x,t) = -v_x \, \frac{\partial}{\partial x}\left[\psi_x^*(x,t)\ \psi_x(x,t)\right] = -\frac{\partial}{\partial x}\left[v_x\ \rho_x(x,t)\right] = -\frac{\partial}{\partial x}j(x,t) \tag{80}$$

or

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) \ = \ - \nabla_{\mathbf{r}}\,\mathbf{j}(\mathbf{r},t) \qquad with \qquad \mathbf{j}(\mathbf{r},t) = \mathbf{v}\ \psi^*(\mathbf{r},t)\ \psi(\mathbf{r},t) \tag{81}$$

where $\mathbf{j}(\mathbf{r},t)$ is the mass-current probability density.

9 The wave equation for relativistic speeds.

We start with the wave eq. (12) from sec. 3

$$\psi_x(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \chi_x(m_{rel_x})\ exp\left[i\frac{c}{\hbar}\left(m_{rel_x}\ x - \ p_x(m_{rel_x})\ t\right)\right] dm_{rel_x} \tag{82}$$

and analyze the equation for relativistic speeds where $\Delta v = c - v \ll c$. We get

$$E_{rel} = E_p = p\ c = \frac{m\ v}{\beta}\ c \qquad \beta = \sqrt{1 - \frac{v^2}{c^2}} \qquad \lambda = \frac{h}{p} \tag{83}$$

The resulting wave equation is

$$\psi_x(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \chi_x(m_{rel_x})\ exp\left[\frac{i}{\hbar}\left(p\ x - \ E_{pv}\ t\right)\right] dm_{rel_x} \tag{84}$$

where

$$E_{pv} = p\ v = \frac{m\ v}{\beta}\ v \tag{85}$$

With $E_{rel} = pc^2/v$ and $E_o^2 \ll E_p^2$ we get

$$E_{pv} = p\ v = \frac{p^2\ c^2}{E_{rel}} = \frac{p^2\ c^2}{\sqrt{E_o^2 + E_p^2}} \approx pc = E_p \tag{86}$$

We now derive the wave equation one time versus space and one time versus time and connect the results with $E_{pv} = pc$. We get

$$\frac{\partial}{\partial t}\psi_x = -\ c\ \frac{\partial}{\partial x}\psi_x \tag{87}$$

10 Applications of the non-relativistic differential equation

The solutions of the time independent non-relativistic differential equation (31) for a potential pot, the harmonic oscillator and the hydrogen atom are derived.

10.1 Potential pot

The non-relativistic time independent differential equation is

$$- i \hbar c \frac{\partial}{\partial x} \psi_x(x) \ + \ U_x(x) \ \psi_x(x) = [E_{tot} \ + \ E_o] \ \psi_x(x) = E \ \psi_x(x) \tag{88}$$

With $y = \psi_x(x)$ we can write

$$- i \hbar c \frac{dy}{y} = [E - U] \ dx \tag{89}$$

After integration we get

$$- i \hbar c \ [\ln |y| + \ln C_y] = \int [E - U] \ dx \tag{90}$$

resulting

$$|y| = \frac{1}{C_y} \ exp \left\{ \frac{i}{\hbar c} \int [E - U] \ dx \right\} \tag{91}$$

Equation (91) is valid for all potential energies U and gives real values for y if

$$\left\{ \frac{i}{\hbar c} \int [E - U] \ dx \right\} = k \pi \qquad and \qquad k = 0, \ \pm 1, \ \pm 2, \ \pm 3, \cdots \tag{92}$$

defining the quantization condition, which together with the normalization condition allows the calculation of the eigenfunctions.

The potential pot is defined as

$$U = \begin{cases} \infty & \text{for } x \leq 0 \\ 0 & \text{for } 0 < x < a \\ \infty & \text{for } x \geq a \end{cases}$$

and we have for $U = 0$

$$\frac{1}{\hbar c} E x = k \pi \qquad resulting \ with \ x = a \qquad E_k = \pi \frac{\hbar c}{a} k \tag{93}$$

with $k = 0, \ \pm 1, \ \pm 2, \ \pm 3, \cdots$.

The total energy is with $E_k = E_{tot} + E_o$

$$E_{tot} = E_k - E_o = \pi \, \frac{\hbar \, c}{a} \, k \; - \; E_o \tag{94}$$

and for $E_{tot} = 0$ we get

$$a_o = k \, \frac{\pi \hbar \, c}{E_o} = k \, \pi \, r_o \qquad with \qquad \frac{\hbar \, c}{E_o} = r_o \tag{95}$$

the radius of of a rest electron or positron.
The eigenfunction is

$$y_k = \frac{1}{C_y} \, exp \left\{ \frac{i}{\hbar \, c} \, E_k \, x \right\} \tag{96}$$

The integration constant C_y we get with the normalization condition

$$\int_{-\infty}^{\infty} y^*_{k'} \; y_k \; dx = \delta_{(k',k)} \tag{97}$$

For $k^{'} = k$ we get

$$\frac{1}{C_y^2} \int_0^a exp \left\{ \frac{i}{\hbar \, c} \, [E_{k'} - E_k] \, x \right\} \; dx = 1 \tag{98}$$

resulting

$$\frac{1}{C_y^2} = a \qquad or \qquad C_y = \sqrt{a} \tag{99}$$

The normalized eigenfunction is

$$y_k = \frac{1}{\sqrt{a}} \, exp \left\{ \frac{i}{\hbar \, c} \, E_k \, x \right\} \tag{100}$$

The main differences compared with the solution obtained with the Schroedinger equation is that the quantization of the energy is proportional to k instead of k^2 and for defined values of a the total energy E_{tot} becomes zero.

10.2 Harmonic oscillator

The potential energy for the harmonic oscillator is

$$U(x) = \frac{K}{2} \, x^2 = \frac{m \, \omega^2}{2} \, x^2 \qquad with \qquad \omega^2 = K/m \tag{101}$$

With eq. (91) we get

$$|y| = \frac{1}{C_y} \, exp \left\{ \, \frac{i}{\hbar \, c} \int \left[E - \frac{K}{2} \, x^2 \right] \, dx \right\} \tag{102}$$

With the quantization condition we get

$$\frac{1}{\hbar\, c} \int_0^a \left[E - \frac{K}{2}\, x^2 \right] \, dx = \frac{1}{\hbar\, c} \, \left[E\, a - \frac{K}{6}\, a^3 \right] = k\, \pi \tag{103}$$

resulting for the quantized energy with $E_{tot} = E_k - E_o$

$$E_{tot} = \pi \, \frac{\hbar\, c}{a} \, \left[k \, + \, \frac{1}{6} \, \frac{m\, \omega^2}{\pi\, \hbar\, c} \, a^3 \right] - E_o \tag{104}$$

The minimum quantum change between two adjacent energy levels is

$$\Delta E_{tot} = \pi \, \frac{\hbar\, c}{a} \tag{105}$$

For $E_{tot} = 0$ we get

$$a \, \left[E_o - \frac{1}{6} \, m\, \omega^2 \, a^2 \right] = k\, \pi\, \hbar\, c \tag{106}$$

which for $k = 0$ gives

$$a_1 = 0 \qquad or \qquad a_{2,3} = \pm \sqrt{\frac{6\, E_o}{m\, \omega^2}} \qquad for \qquad k = 0 \tag{107}$$

With the normalization condition given by equation (97) we have that

$$\int_{-\infty}^{\infty} y_{k'}^{*} \; y_k \; dx = \frac{1}{C_y^2} \, \int_{-\infty}^{\infty} exp \left\{ \frac{i}{\hbar\, c} \, [E_{k'} - E_k] \; x \right\} \; dx \tag{108}$$

or

$$\frac{\hbar\, c}{C_y^2} \, \int_{-\infty}^{\infty} exp \{ \, i \, [E_{k'} - E_k] \; \eta \} \; d\eta = \frac{\hbar\, c}{C_y^2} \; \delta_{(k',k)} \quad with \quad \eta = \frac{x}{\hbar\, c} \tag{109}$$

With $k^{'} = k$ we get the integration constant $C_y = \sqrt{\hbar\, c}$ resulting the normalized eigenfunctions

$$y_k = \frac{1}{\sqrt{\hbar\, c}} \, exp \left\{ \, \frac{i}{\hbar\, c} \, \left[E_k \; x - \frac{K}{6} \; x^3 \right] \right\} \tag{110}$$

The main differences compared with the solution obtained with the Schroedinger equation is, that the minimum quantum energy change between two adjacent energy levels is constant and independent of the oscillation frequency ω and, that for defined values of a the total energy E_{tot} becomes zero.

10.3 Hydrogen atom

We start with the deduction of the quantization conditions with eq. (31)

$$-\,i\,\hbar\,c\,\frac{\partial}{\partial x}\psi_x(x)\;+\;U_x(x)\;\psi_x(x)=[E_{tot}\;+\;E_o]\;\psi_x(x)=E\;\psi_x(x) \tag{111}$$

We define the operator

$$\nabla\psi(x,y,z)=\frac{\partial}{\partial x}\psi(x,y,z)\;+\;\frac{\partial}{\partial y}\psi(x,y,z)\;+\;\frac{\partial}{\partial z}\psi(x,y,z) \tag{112}$$

which is the operator for the divergence normally applied on separate functions for each coordinate. To differentiate the symbol now used from the same symbol used for the Hamilton operator, we propose to use for the Hamilton operator the symbol $\vec{\nabla}$ to show its vector character.

For polar coordinates we write

$$-\,i\,\hbar\,c\,\nabla\;\chi(r,\theta,\varphi)\;+\;U\;\chi(r,\theta,\varphi)=E\;\chi(r,\theta,\varphi) \tag{113}$$

with the ∇ operator expressed in polar coordinates

$$\nabla=\frac{\partial}{\partial r}\;+\;\frac{2}{r}\;+\;\frac{1}{r\;\sin\theta}\,\frac{\partial}{\partial\varphi}\;+\;\frac{1}{r}\,\frac{\partial}{\partial\theta}\;+\;\frac{1}{r}\;\cot\theta \tag{114}$$

The differential equation has now the form

$$\left[\nabla\;+\;\frac{i}{\hbar\,c}\,U\right]\,\chi=\frac{i}{\hbar\,c}\,E\,\chi \tag{115}$$

We now assume that the wave function χ can be expressed as a product of a function exclusively of the distance r and a function of the angular variables θ and φ.

$$\chi(r,\theta,\varphi)=R(r)\;Y(\theta,\varphi) \tag{116}$$

We get

$$\left[\frac{d}{r}+\frac{4}{r}\right]R\cdot Y+\frac{1}{r}\,\Lambda Y\cdot R+\frac{i}{\hbar\,c}U\cdot R\cdot Y=\frac{i}{\hbar\,c}E\cdot R\cdot Y \tag{117}$$

with the operator Λ

$$\Lambda=\frac{1}{\sin\theta}\,\frac{\partial}{\partial\varphi}\;+\;\frac{\partial}{\partial\theta}\;+\;2\;\cot\theta \tag{118}$$

We now assume that

$$\Lambda Y=-\lambda Y \tag{119}$$

and get two separate differential equations for $R(r)$ and $Y(\theta, \varphi)$.

$$\frac{d}{r} R - \frac{i}{\hbar\, c} [E - U]\, R + \frac{1}{r} [4 - \lambda]\, R = 0 \tag{120}$$

and

$$\left[\frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} \; + \; \frac{\partial}{\partial\theta} \; + \; 2 \; \cot\theta \right] Y = -\lambda Y \tag{121}$$

Eq. (120) gives

$$\ln R = \frac{i}{\hbar\, c} \int_{r_o}^{r} [E - U]\, dr - [4 - \lambda] \ln \frac{r}{r_o} \; + \; C_R \tag{122}$$

with $C_R = C_r + i\, C_i$ a complex integration constant.

From the solution of eq. (121) results that $\lambda = i\, l$ with $l = \; 0, \; \pm 1, \; \pm 2; \cdots$ as will be shown later on. We get

$$R = \; exp \left\{ -4 \; \ln \frac{r}{r_o} + C_r \right\} \; exp \left\{ \frac{i}{\hbar\, c} \left[\left[\int_{r_o}^{r} (E - U) dr + l\, \hbar\, c \; \ln \frac{r}{r_o} + C_i\, \hbar\, c \right] \right] \right\} \tag{123}$$

The quantization condition requires that

$$\frac{1}{\hbar\, c} \left[\int_{r_o}^{r} (E - U) dr + l\, \hbar\, c \; \ln \frac{r}{r_o} + C_i\, \hbar\, c \right] = k\, \pi \quad with \quad k = \; 0, \; \pm 1, \; \pm 2; \cdots \tag{124}$$

Equation (124) is valid for all point symmetrical potentials U. We now introduce the potential of an atomic nucleus

$$U = \; - \; \frac{Z\, e^2}{4\pi\, \epsilon_o\, r} = \; - \; \frac{K_u}{r} \qquad with \qquad K_u = \frac{Z\, e^2}{4\pi\, \epsilon_o} \tag{125}$$

where Z is the atomic number, and get for the quantization condition with $E = E_k$

$$E_k = \left[\; k\, \pi\, \hbar\, c - (K_u + l\, \hbar\, c) \; \ln \frac{r}{r_o} - C_i\, \hbar\, c \right] \; \frac{1}{r} \tag{126}$$

or

$$\ln \frac{r}{r_o} = \left(k\, \pi - \frac{E_k\, r}{\hbar\, c} - C_i \right) / \left(\frac{K_u}{\hbar\, c} + l \right) \tag{127}$$

The energy E_k of eq. (126) must be equal to one term of the series of the hydrogen spectrum empirically introduced by Ballmer, namely

$$E_n = h\; c\; R_H \; \frac{1}{n^2} \qquad and \qquad \Delta E_n = \; c\; R_H \; \left[\frac{1}{n^2} - \frac{1}{(n+1)^2} \right] \tag{128}$$

with R_H the Rydberg constant and $n = 1,\ 2, ...$

This is only possible if the product $E_k\ r$ in eq. (127) is quantized. We define that $E_k\ r = A_q(k,l)$ and get

$$E_k = \frac{A_q}{r_o}\ exp\left\{ -\ \left(k\ \pi - \frac{A_q}{\hbar\ c} - C_i\right) / \left(\frac{K_u}{\hbar\ c} + l\right)\right\} \tag{129}$$

The problem reduces now to find the function $A_q(k,l)$ that makes $E_k = E_{tot} + E_0$ with $E_{tot} = E_n$ from the hydrogen spectrum.

Before we continue we will deduce the condition $\lambda = i\ l$ introduced previously. We assume that

$$Y(\theta,\phi) = \Theta(\theta)\ \Phi(\varphi) \qquad and \qquad \frac{d}{d\varphi}\Phi = m\ \Phi \tag{130}$$

and with $\Phi(\varphi) = \Phi(\varphi + 2\pi)$ we get

$$\Phi = exp\{m\ \varphi\} \qquad with \qquad m = i\ m_l \qquad and \qquad m_l = \pm 0,\ \pm 1,\ \pm 2; \cdots \tag{131}$$

With eq. (130) we have that eq. (121) transforms to

$$\frac{m}{\sin\theta}\ \ \Theta + \frac{d}{d\theta}\Theta\ +\ 2\ \cot\theta\ \Theta = -\ \lambda\ \Theta \tag{132}$$

and

$$\frac{d\Theta}{\Theta} = -\ \left[\ \frac{m}{\sin\theta}\ +\ 2\ \cot\theta\ +\ \lambda\right]\ d\theta \tag{133}$$

which gives the solution

$$\Theta = \frac{1}{C_\Theta}\ exp\left\{-\int\left[\ \frac{i\ m_l}{\sin\theta}\ +\ 2\ \cot\theta\ +\ \lambda\right] d\theta\ \right\} \tag{134}$$

With $\Theta(\theta) = \Theta(\theta + 2\pi)$ we conclude that

$$\Theta = \frac{1}{C_\Theta}\ exp\{-2\ln\sin\theta\}\ \ exp\{-i\ \ [\ m_l\ln(\csc\theta - \cot\theta)\ +\ l\ \theta\ ]\} \tag{135}$$

with $\lambda = i\ l$ and $l = \pm 0,\ \pm 1,\ \pm 2; \cdots$ what we have anticipated for eq. (123).

Eq.(133) we can now write as

$$\frac{d}{d\theta}\Theta\ +\ i\ \frac{m_l}{\sin\theta}\ \ \Theta\ =\ -\ 2\ \cot\theta\ \Theta\ -\ i\ l\ \Theta \tag{136}$$

In this equation the real and the imaginary terms must be equal, and we get from the imaginary terms that

$$\frac{m_l}{l} = -\sin\theta \qquad with \qquad m_l = \pm 0,\ \pm 1,\ \pm 2; \cdots \quad and \quad l = \pm 0,\ \pm 1,\ \pm 2; \cdots \tag{137}$$

We conclude, that the relation between the orbital quantum number l and the magnetic quantum number m_l is

$$\left|\frac{m_l}{l}\right| \leq 1 \qquad or \qquad |m_l| \ \leq \ |l| \tag{138}$$

Now we have to find $A_q(k,l)$ that makes $E_k = E_n + E_o$.

10.3.1 Calculations made for the Hydrogen atom

A numerical calculation is required to obtain $E_k\ r = A_q(k,l)$ from eq. (129)

$$E_k = \frac{A_q}{r_o}\ exp\left\{ -\ \left(k\ \pi - \frac{A_q}{\hbar\ c} - C_i\right) / \left(\frac{K_u}{\hbar\ c} + l\right)\right\} \quad with \quad E_k = E_{tot} + E_o \tag{139}$$

and the help of

$$E_{tot} = E_n \qquad and \qquad E_k = -(h\ c\ R_H \frac{1}{k^2} + E_o) \tag{140}$$

where the sign for E_k was changed because eq. (139) gives only acceptable results with $r > 0$ if $E_k\ r = A_q(k,l) < 0$. That is because the external potential U was defined negative with $U \rightarrow 0$ for $r \rightarrow \infty$. After obtaining A_q we can calculate the radius $r = A_q/E_k$. The calculations were made with $l = 1$ and with $C_i = 500$.

If $E_k\ r = A_q(k,l)$ could be express analytically it would be possible to calculate the differences $\Delta E_n = \Delta E_k$ directly with

$$E_k(r,k,l,Ci) = \left[\ k\ \pi\ \hbar\ c - (K_u + l\ \hbar\ c)\ \ln\frac{r}{r_o} - C_i\ \hbar\ c\right]\ \frac{1}{r} \tag{141}$$

end the help of

$$r(k,l,Ci) = r_o\ exp\ \left\{\left(k\ \pi - \frac{A_q}{\hbar\ c} - C_i\right) / \left(\frac{K_u}{\hbar\ c} + l\right)\ \right\} \tag{142}$$

which was derived from eq. (127).

Results of calculations:

- The energy E_k tends to the negative Energy of a rest electron instead to zero for $k \to \infty$ as it is for the Rydberg term (see Fig. 1). The energy-radius product A_q is shown in Fig. 2.

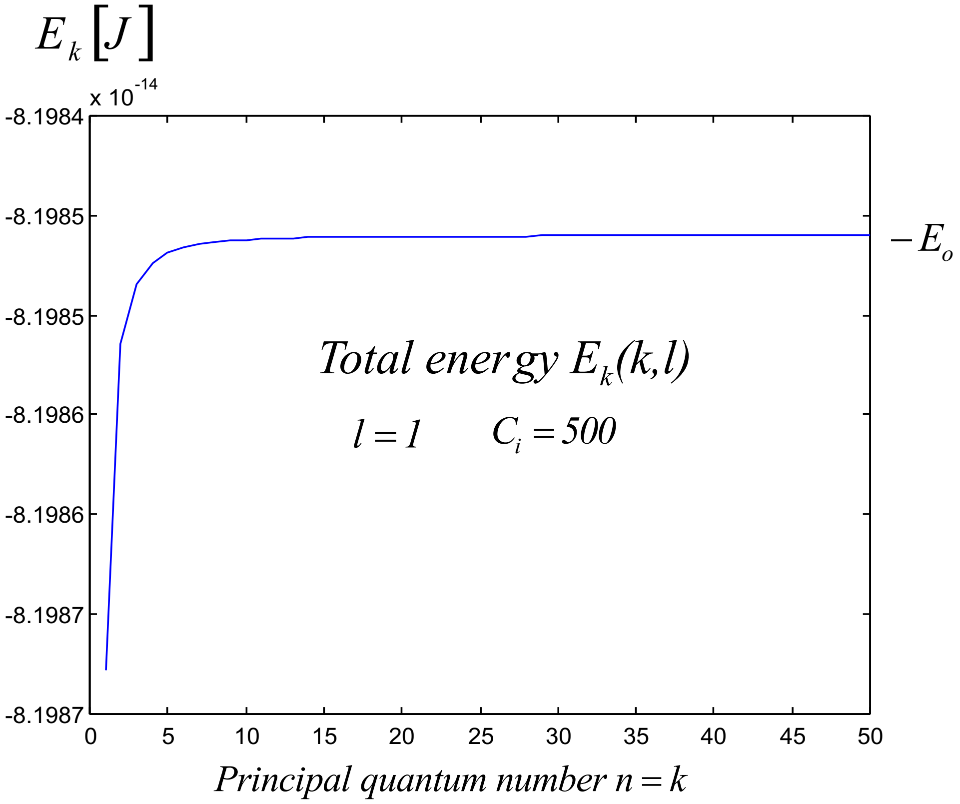

Figure 1: Energy E_k of Hydrogen atom

- The radius of the Hydrogen atom decreases with increasing energy E_k for constant orbital number l (see Fig.3). This is because for a constant orbital number l the energy can only increase by contracting the orbital radius r, increasing the tangential speed of the electron. The orbital radius r is of the order of $10^{-10}m$ with $l = 1$, which is also the order of magnitude of the inter-atomic distances and the Bohr radius $a_o \approx 0.5 \overset{o}{A}$

- The radius increases with constant energy E_k or constant n and increasing orbital quantum number l (see Fig.4). The tangential speed of the electron decreases with the increasing radius r.

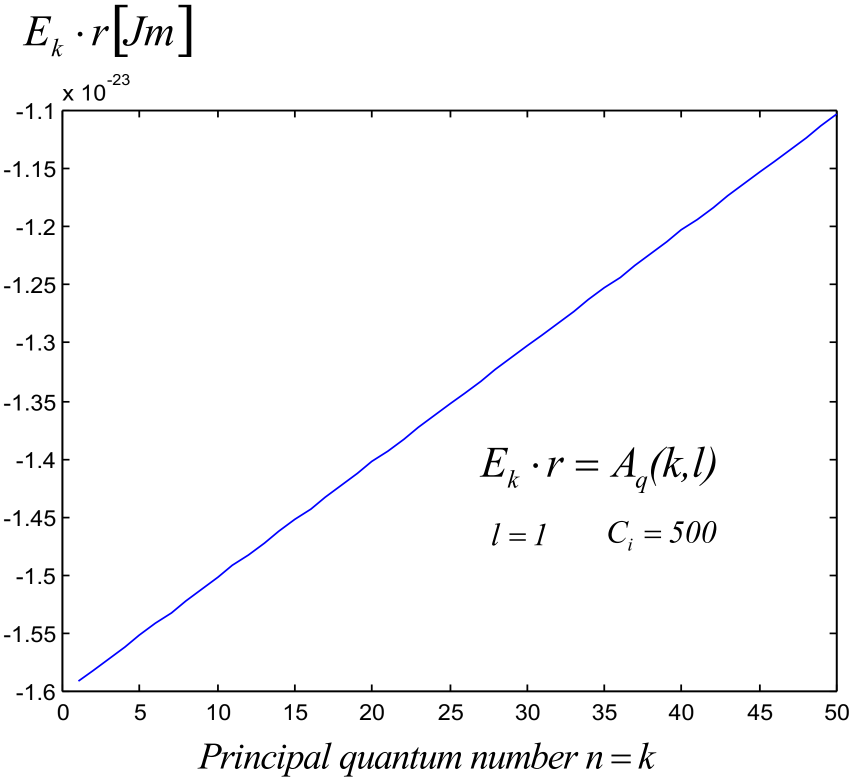

Figure 2: Energy-radius product Aq of Hydrogen atom

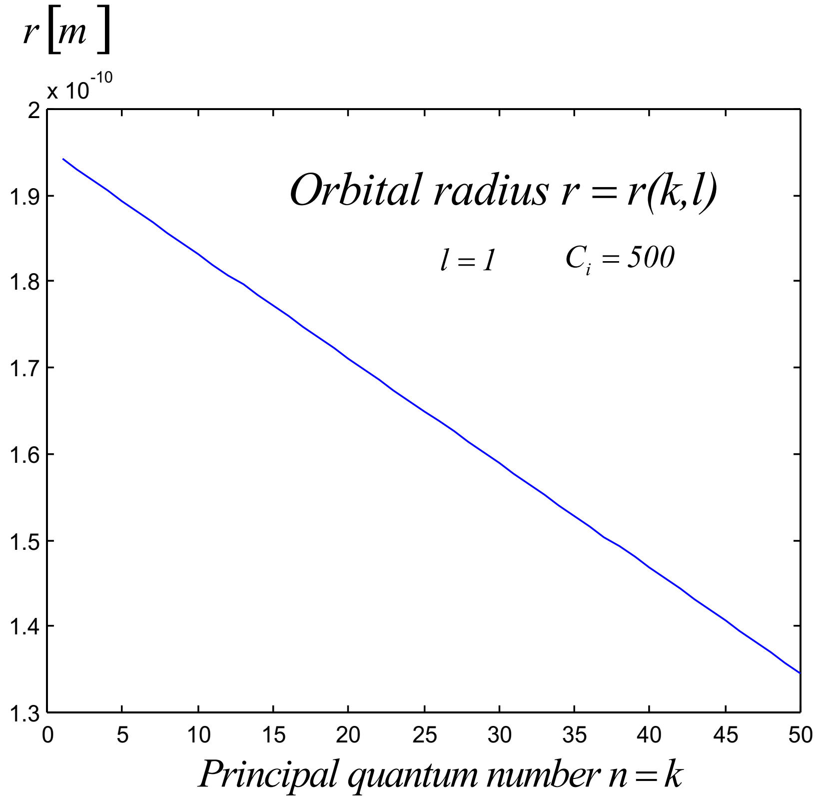

Figure 3: Orbital radius r of Hydrogen atom for constant $l = 1$.

- The energy eigenvalues don't depend exclusively from the principal quantum number $n = k$. There is no degeneracy of the principal quantum state n, because the eigenvalues depend also from the orbital quantum number l. It is important to note, that the principal quantum state n in the Hydrogen solution of the Schroedinger equation is also a function of the orbital quantum number l because it was defined as $n = n_r + l + 1$.

- The proposed theoretical approach explains the two spin states with two types of electrons and positrons, namely, the accelerating and decelerating electrons and positrons. The fine-structure splitting of the Hydrogen spectrum is thus produced by the orbital quantum state l, the finite mass of the nucleus and probably by relativistic effects, and not by the supposed magnetic spin moment attributed to the electron by standard theory.

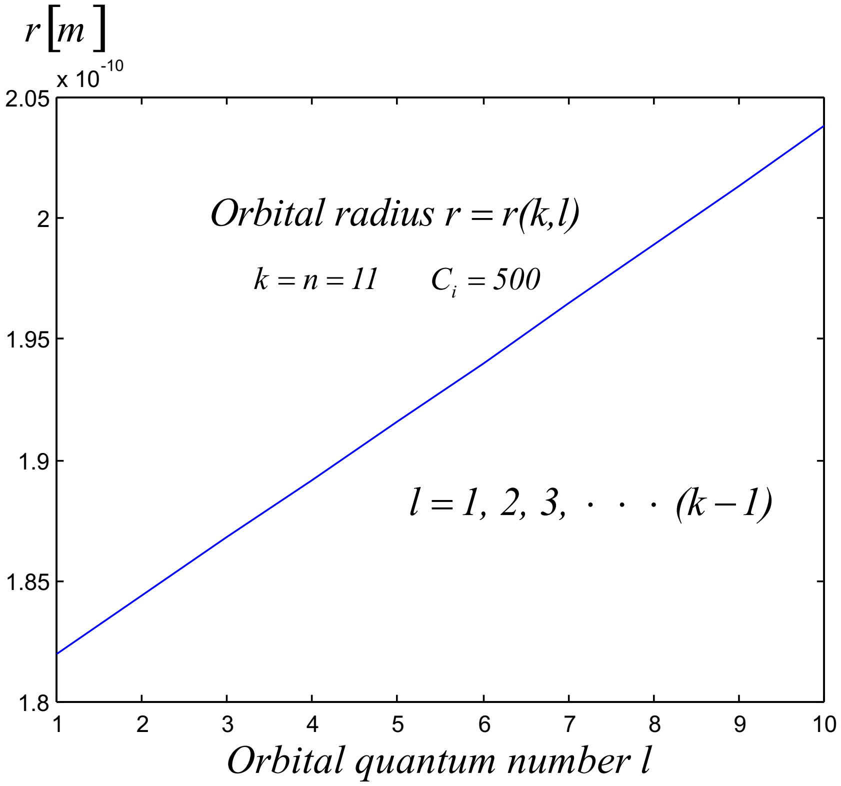

Figure 4: Orbital radius r of Hydrogen atom for constant $k = 11$.

11 Conclusions

The fact that it is possible to derive with the proposed theory the time independent **Schroedinger** differential equation (74), ensures that the proposed approach is in accordance with the correspondence principle of the quantum mechanics. Accordingly, results of solving the time independent **Schroedinger** differential equation (eg. the harmonic oscillator, the radial Coulomb potential or hydrogen atom, the movement of a relativistic particle deduced by Dirac, etc.), are also valid for the proposed theory.

The following main characterisitics were identified,

- The uncertainty principle of Heisenberg between canonical conjugated variables links the pairs **energy-space** and **momentum-time**.
- Because of the changed canonical conjugated variables, the relativistic and non-relativistic operators are reversed respect to time and space derivations compared with standard theory.
- The Schrödinger equation is replaced by an equation where the wave function is differentiated one time versus space and two times versus time in analogy to Newton's second law.
- The time independent **Schrödinger** equation is obtained by deriving the new wave funktion two times versus space, the same as for the established wave function.
- In applying the relativistic operators to the relativistic energy of a particle the **Klein-Gordon** differential equation results.
- The **Hamilton** operator remains formally unchanged.

12 References.

1. Max Schubert / Gerhard Weber. **Quantentheorie, Grundlagen und Anwendungen.** Spektrum, Akad. Verlag 1993.
2. Benenson · Harris · Stocker · Lutz. **Handbook of Physics.** Springer Verlag 2001.
3. Günter Bärwolff. **Höhere Mathematik.** 2.Auflage 2006. Elsevier Spektrum Akademischer Verlag.

4. O. Domann. **Definition of a space with fundamental particles with longitudinal and transversal rotational momentum, and deduction of the linear momentum which generates electrostatic, magnetic, induction and gravitational forces.** June 2003. http://www.odomann.com/